The Suspension Calculus

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Outline

1. Using the Lambda Calculus for Representation
2. The Suspension Calculus
3. Other Explicit Substitution Calculi
4. Contributions and Future Work

Lambda Calculus as a Representational Device

Abstraction in the lambda calculus can capture binding in syntactic objects

Example

The formula

$$\forall x.(P(x) \vee Q)$$

can be encoded as

$$\mathit{forall}\ (\lambda x.\ (\mathit{or}\ (P\ x)\ Q))$$

Example

The expression

$$((\lambda x.\ x + 1)\ 2)$$

can be encoded as

$$\mathit{app}\ (\mathit{abs}\ \lambda x.\ (\mathit{plus}\ x\ (\mathit{const}\ 1)))\ (\mathit{const}\ 2)$$

Benefits of Such a Representation

Variable Renaming

$\forall y.(p(y) \vee q)$ is automatically equivalent to $\forall z.(p(z) \vee q)$

Quantifier Instantiation

$(\mathit{forall}\ P) \xrightarrow{t} (P\ t)$

Sophisticated Pattern Matching

We encode the pattern

$$\forall x.(P(x) \vee Q)$$

as

$$\mathit{forall}\ (\lambda x.\ (\mathit{or}\ (P\ x)\ Q))$$

which captures the notion that P can contain x but Q cannot

The De Bruijn Representation of Lambda Terms

Key Idea

Instead of using names to associate variable occurrences with their binders, we count the number of abstractions between a variable occurrence and its binder

Example

We represent the lambda term

$$(\lambda x.\ (\lambda y.\ x\ y)\ x)$$

by the de Bruijn term

$$(\lambda\,(\lambda\,\#2\ \#1)\ \#1)$$

In this representation α-convertible terms are identical

Explicit Substitutions and β-reduction

Key Idea

Laziness in substitution is important to implementations

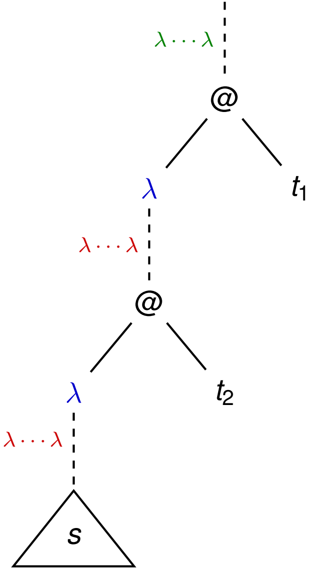
- Sometimes substitution can be avoided altogether, *e.g.*

$$(\lambda x.c\ t_1)\ t_2 \stackrel{?}{=} d\ t_3$$

- Laziness is the basis for sharing substitution walks, *e.g.*

$$(\lambda x.\lambda y.t_1)\ t_2\ t_3$$

The General Scenario to be Treated



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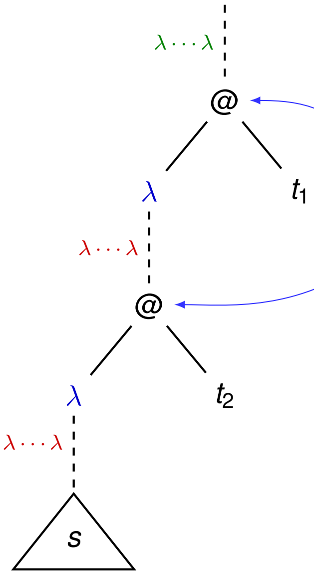

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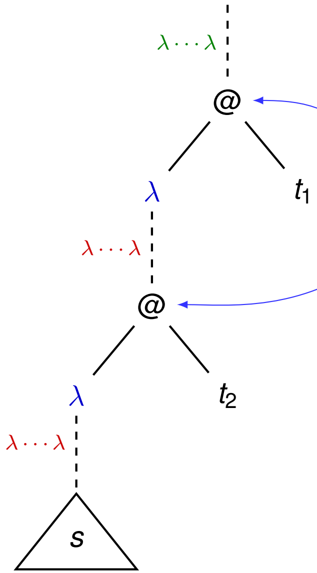

We want to contract these redexes but delay their effect on s

The result will have the form $[\![s, ol, nl, e]\!]$

Syntax of the Suspension Calculus

An explicit treatment of substitutions is gained by adding a new term of the form $[\![t, ol, nl, e]\!]$ where

- t is a term whose skeleton we substitute over
- ol is the old embedding level of t
- nl is the new embedding level of t
- e is an environment of substitutions of the form

$$(t_1, l_1) :: (t_2, l_2) :: \ldots :: (t_n, l_n) :: nil$$

where t_i is the substitution for the index $\#i$ and l_i is its embedding level

A Simple Rewriting Calculus

$$(\beta_s) \qquad ((\lambda\, t_1)\ t_2) \rightarrow [\![t_1, 1, 0, (t_2, 0) :: nil]\!]$$

A Simple Rewriting Calculus

(β_s) $((\lambda\, t_1)\ t_2) \to [\![t_1, 1, 0, (t_2, 0) :: nil]\!]$

(r1) $[\![(t_1\ t_2), ol, nl, e]\!] \to ([\![t_1, ol, nl, e]\!]\ [\![t_2, ol, nl, e]\!])$

(r2) $[\![(\lambda\, t), ol, nl, e]\!] \to (\lambda\, [\![t, ol', nl', (\#1, nl') :: e]\!])$,
where $ol' = ol + 1$ and $nl' = nl + 1$

(r3) $[\![\#1, ol, nl, (t, l) :: e]\!] \to [\![t, 0, nl', nil]\!]$, where $nl' = nl - l$

(r4) $[\![\#i, ol, nl, (t, l) :: e]\!] \to [\![\#i', ol', nl, e]\!]$,
where $i' = i - 1$ and $ol' = ol - 1$, provided $i > 1$

(r5) $[\![\#i, 0, nl, nil]\!] \to \#j$, where $j = i + nl$

(r6) $[\![c, ol, nl, e]\!] \to c$, provided c is a constant

A Simple Example

$$(\lambda\,(\lambda\,(\#1\ \#2)))\ t$$

$$\rhd_{\beta_s}\ [\![\lambda\,(\#1\ \#2), 1, 0, (t, 0) :: nil]\!]$$

$$\rhd_{r2}\ \lambda\,[\![(\#1\ \#2), 2, 1, (\#1, 1) :: (t, 0) :: nil]\!]$$

$$\rhd_{r1}\ \lambda\,([\![\#1, 2, 1, (\#1, 1) :: (t, 0) :: nil]\!]\ [\![\#2, 2, 1, (\#1, 1) :: (t, 0) :: nil]\!])$$

$$\rhd_{r3}\ \lambda\,([\![\#1, 0, 0, nil]\!]\ [\![\#2, 2, 1, (\#1, 1) :: (t, 0) :: nil]\!])$$

$$\rhd_{r5}\ \lambda\,(\#1\ [\![\#2, 2, 1, (\#1, 1) :: (t, 0) :: nil]\!])$$

$$\rhd_{r4}\ \lambda\,(\#1\ [\![\#1, 1, 1, (t, 0) :: nil]\!])$$

$$\rhd_{r3}\ \lambda\,(\#1\ [\![t, 0, 1, nil]\!])$$

Motivation for Merging

We have a mechanism for multiple non-trivial substitutions, but our system doesn't yet have them

Example

The term

$$(\lambda\,\lambda\,t_1)\ t_2\ t_3$$

reduces to

$$[\![[\![t_1, 2, 1, (\#1, 1) :: (t_2, 0) :: nil]\!], 1, 0, (t_3, 0) :: nil]\!]$$

but no rule applies to the outer substitution

In order to merge these two we need to generate the merging of two environments

Rules for Merging Environments

(m1) $[\![[\![t, ol_1, nl_1, e_1]\!], ol_2, nl_2, e_2]\!] \to [\![t, ol', nl', \{\!\{e_1, nl_1, ol_2, e_2\}\!\}]\!]$,
where $ol' = ol_1 + (ol_2 \dot{-} nl_1)$ and $nl' = nl_2 + (nl_1 \dot{-} ol_2)$

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where $ol' = ol_1 + (ol_2 \dot{-} nl_1)$ and $nl' = nl_2 + (nl_1 \dot{-} ol_2)$

(m2) $\{\!\{ e_1, nl_1, 0, nil \}\!\} \rightarrow e_1$

(m3) $\{\!\{ nil, 0, ol_2, e_2 \}\!\} \rightarrow e_2$

(m4) $\{\!\{ nil, nl_1, ol_2, (t, l) :: e_2 \}\!\} \rightarrow \{\!\{ nil, nl_1', ol_2', e_2 \}\!\}$,
where $nl_1' = nl_1 - 1$ and $ol_2' = ol_2 - 1$, provided $nl_1 \geq 1$

(m5) $\{\!\{ (t, n) :: e_1, nl_1, ol_2, (s, l) :: e_2 \}\!\} \rightarrow \{\!\{ (t, n) :: e_1, nl_1', ol_2', e_2 \}\!\}$,
where $nl_1' = nl_1 - 1$ and $ol_2' = ol_2 - 1$, provided $nl_1 > n$

(m6) $\{\!\{ (t, n) :: e_1, n, ol_2, (s, l) :: e_2 \}\!\} \rightarrow$
$([\![t, ol_2, l, (s, l) :: e_2]\!], m) :: \{\!\{ e_1, n, ol_2, (s, l) :: e_2 \}\!\}$,
where $m = l + (n \dot{-} ol_2)$

Properties of the Suspension Calculus

Theorem

The reading and merging rules define a terminating and confluent system

Proof structure:

- Termination used an extended recursive path ordering [Der82, FZ95] — I also verified this using Coq
- Confluence used weak confluence and termination

Theorem

The full system is confluent

Proof structure: Used the technique from [CHL96]

Graftable Meta Variables and Confluence

For X a graftable meta variable, $[\![X, ol, nl, e]\!]$ is irreducible which makes confluence an issue

Example

$((\lambda\,((\lambda\,X)\ t_1))\ t_2)$ can be rewritten to either of the following

$$[\![[\![X, 1, 0, (t_1, 0) :: nil]\!], 1, 0, (t_2, 0) :: nil]\!]$$

$$[\![[\![X, 2, 1, (\#1, 1) :: (t_2, 0) :: nil]\!], 1, 0, (t_1', 0) :: nil]\!]$$

where $t_1' = [\![t_1, 1, 0, (t_2, 0) :: nil]\!]$

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The reading rules do not suffice to ensure a common reduct

But the merging rules guarantee one exists

Comparison at Two Levels

1. Property based comparison

	Combination	Confluence	PSN
Suspension calculus	Yes	Yes	?
$\lambda\sigma$-calculus	Yes	Yes	No
λs-calculus	No	No	Yes
λs_e-calculus	No	Yes	No
λ_{ws}-calculus	No	Yes	Yes

2. Behavior based comparison
   - Describe information preserving translations
   - Translations provide insight into behavior

Contributions

- A modified version of the suspension calculus
  - Merging rules which are usable in practice
  - Structure for composition which retains logical properties
  - New proofs for termination and confluence
- A comparison of explicit substitution calculi
  - Translations between calculi
  - Proofs of formal properties of the translations

Future Work

- Preservation of strong normalization
- New methods of higher-order unification
- Compilation of strong reduction